Appendix 3.1

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1. Differentiation of $\sin x$. Looking back at Example 3.1.5 we find

$$\lim_{x\to a} \frac{\sin x - \sin a}{x-a} = \cos a \lim_{h\to 0} \frac{\sin h}{h} + \sin a \lim_{h\to 0} \frac{\cos h - 1}{h}. \tag{6}$$

So we need to evaluate

$$\lim_{h\to 0} \frac{\sin h}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{\cos h - 1}{h}.$$

This was done in a previous section by considering the areas of triangles and sectors of circles. You might have been tempted, instead, to use L'Hôpital's Rule. For example

$$\lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\cos h}{1} = 1.$$

Yet to do this you need to know that the derivative of $\sin x$ is $\cos x$. This would lead to

$$\frac{d}{dx}\sin x = \cos x \overset{\text{L'Hôpital}}{\Longrightarrow} \lim_{h\to 0}\frac{\sin h}{h} = 1 \overset{\text{by (6)}}{\Longrightarrow} \frac{d}{dx}\sin x = \cos x.$$

A classic example of a circular argument.

2. **Example 3.1.16** *Prove the Sum Rule for derivatives.*

Solution: Consider

$$\begin{aligned}
\lim_{x\to a}\frac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a}\frac{f(x)+g(x)-f(a)-g(a)}{x-a} \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\lim_{x\to a}\frac{g(x)-g(a)}{x-a} \\
&\qquad \text{by the Sum Rule for Limits,} \\
&= f'(a)+g'(a).
\end{aligned}$$

Since these last two limits exist we justify the use of the Sum Rule for limits as well as proving that $f+g$ is differentiable at a. Further

$$(f+g)'(a)=f'(a)+g'(a).$$

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3. Let $n\in\mathbb{N}$ and $a\neq 0$ be given. For $x\neq 0$ consider

$$\frac{\frac{1}{x^n}-\frac{1}{a^n}}{x-a}=-\frac{1}{x^n a^n}\frac{x^n-a^n}{x-a}.$$

So by the rules for limits

$$\begin{aligned}
\lim_{x\to a}\frac{\frac{1}{x^n}-\frac{1}{a^n}}{x-a} &= -\frac{1}{(\lim_{x\to a}x^n)\,a^n}\lim_{x\to a}\frac{x^n-a^n}{x-a} \\
&= -\frac{1}{a^{2n}}na^{n-1} \quad \text{by the work done on } \frac{dx^n}{dx}, \\
&= -na^{-n-1}.
\end{aligned}$$

So

$$\left.\frac{dx^{-n}}{dx}\right|_{x=a}=-na^{-n-1}.$$

Yet $a\neq 0$ was arbitrary, hence

$$\frac{dx^{-n}}{dx}=-nx^{-(n+1)}$$

for all $x\neq 0$. ■

4. **More Examples** In an earlier Appendix the following functions were seen to be continuous on $\mathbb{R}$. Are they differentiable on $\mathbb{R}$?

$$f_1(x) = \begin{cases} \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \qquad f_2(x) = \begin{cases} \dfrac{\sin\theta}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0. \end{cases}$$

$$f_3(x) = \begin{cases} \dfrac{\cos\theta - 1}{\theta} & \text{if } \theta \neq 0 \\ 0 & \text{if } \theta = 0. \end{cases} \qquad f_4(x) = \begin{cases} \dfrac{\cos\theta - 1}{\theta^2} & \text{if } \theta \neq 0 \\ -\dfrac{1}{2} & \text{if } \theta = 0. \end{cases}$$

$$f_5(x) = \begin{cases} \dfrac{e^x - 1 - x}{x^2} & \text{if } x \neq 0 \\ \dfrac{1}{2} & \text{if } x = 0. \end{cases}$$

By the Quotient Rule they are all differentiable for non-zero x or θ. At 0 we have to return to the definition. For example, for f_1 consider, for $x \neq 0$,

$$\frac{f_1(x) - f_1(0)}{x - 0} = \frac{\frac{e^x-1}{x} - 1}{x} = \frac{e^x - 1 - x}{x^2} = f_5(x)\,.$$

Thus

$$\lim_{x\to 0} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x\to 0} f_5(x) = \frac{1}{2}.$$

Hence f_1 is differentiable at 0 with $f_1'(0) = 1/2$.

For f_2 we would get

$$\frac{f_2(\theta) - f_2(0)}{\theta - 0} = \lim_{\theta\to 0} \frac{\sin\theta - \theta}{\theta^2}.$$

But then, what is

$$\lim_{\theta\to 0} \frac{\sin\theta - \theta}{\theta^2}?$$

There is no elementary way to evaluate this, so we will have to wait for later results.

I leave the other functions to students to consider. ■

5. **Warning** In the proof of the Product Rule

$$(fg)'(a) = f'(a)\,g(a) + f(a)\,g'(a)$$

we started by looking at the LHS, $(fg)'(a)$. Do **not** start by looking at the RHS, for you are likely to write

$$\begin{aligned} f'(a)\,g(a) + f(a)\,g'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}g(a) + f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a}\left(\frac{(f(x)-f(a))\,g(a) + f(a)\,(g(x)-g(a))}{x-a}\right). \end{aligned}$$

Though this is not wrong it is not going to simplify to $(fg)'(a)$.

Similarly, to prove

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g^2(a)},$$

we started by examining the LHS. Do not start by looking at the RHS, for you are likely to write

$$-\frac{g'(a)}{g^2(a)} = -\frac{1}{g^2(a)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{\frac{1}{g(a)} - \frac{g(x)}{g^2(a)}}{x-a}.$$

Again this is not wrong but it is not going to simplify to $(1/g)'(a)$.

6. A **Warning Note** on (3). You might be tempted to replace (3) by

$$\frac{f(g(x))-f(g(k))}{x-k} = \frac{f(g(x))-f(g(k))}{g(x)-g(k)}\left(\frac{g(x)-g(k)}{x-k}\right) \qquad (7)$$

and try to say:

"let $x \to k$ for then $g(x) \to g(k)$ (since g is differentiable at k implies g is continuous at k). Then the right hand side of (7) tends to $f'(g(a))\,g'(a)$, giving the Composite Rule."

But this would be **WRONG**, because (7) only holds when $x \neq a$ **and** $g(x) \neq g(k)$ and it might be the case that $g(x) = g(k)$ for infinitely many x as $x \to k$. This is why the

$$\frac{f(g(x))-f(g(k))}{g(x)-g(k)}$$

in (7) is replaced by $F_\ell(g(x))$ in (3).

7. **Alternative proof of the Composite Rule for differentiation.**

In MATH20132, Calculus of severable variables, this result is generalised to functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. The proof given in the notes does not generalise so I will give one here that does.

The usual definition that $g'(k)$ exists can be written in the form

$$\lim_{t\to 0} \frac{g(k+t) - g(k) - tg'(k)}{t} = 0.$$

Write $r_1(t) = g(k+t) - g(k) - tg'(k)$ so $\lim_{t\to 0} r_1(t)/t = 0$. Similarly, write the definition that $f'(\ell)$ exists in the form

$$\lim_{u\to 0} \frac{f(\ell+u) - f(\ell) - uf'(\ell)}{u} = 0,$$

and let $r_2(u) = f(\ell+u) - f(\ell) - uf'(\ell)$, so $\lim_{u\to 0} r_2(u)/u = 0$.

Our aim is to show that

$$\lim_{w\to 0} \frac{(f\circ g)(k+w) - (f\circ g)(k)}{w} = g'(k)\, f'(\ell)$$

or

$$\lim_{w\to 0} \frac{(f\circ g)(k+w) - (f\circ g)(k) - wg'(k)\, f'(\ell)}{w} = 0.$$

Writing

$$R(w) = (f\circ g)(k+w) - (f\circ g)(k) - wg'(k)\, f'(\ell)$$

the aim becomes to prove that $\lim_{w\to 0} R(w)/w = 0$.

We start with a rearrangement

$$\begin{aligned}
R(w) &= f\big(g(k+w)\big) - f\big(g(k)\big) - wg'(k)\, f'(\ell) \\
&= f\big(r_1(w) + \ell + wg'(k)\big) - f(\ell) - wg'v(k)\, f'(\ell) \\
&\qquad \text{by definition of } r_1 \text{ and } \ell = g(k) \\
&= r_2\big(r_1(w) + wg'(k)\big) + \big(r_1(w) + wg'(k)\big) f'(\ell) - wg'(k)\, f'(\ell) \\
&\qquad \text{by definition of } r_2, \\
&= r_2\big(r_1(w) + wg'(k)\big) + r_1(w)\, f'(\ell)\,.
\end{aligned}$$

Hence
$$\frac{R(w)}{w} = \frac{r_2\big(r_1(w) + wg'(k)\big)}{w} + \frac{r_1(w)}{w} f'(\ell)\,.$$

Go back to $\varepsilon - \delta$ definition of limits to finish the proof.

Let $\varepsilon > 0$ be given.

The definition of $\lim_{w \to 0} r_1(w)/w = 0$ implies there exists $\delta_1 > 0$ such that if $0 < |w| < \delta_1$ then

$$\left|\frac{r_1(w)}{w}\right| < \frac{\varepsilon}{2\,(1 + |f'(\ell)|)} \quad \text{i.e.} \quad \left|\frac{r_1(w)}{w} f'(\ell)\right| < \frac{\varepsilon}{2} \frac{|f'(\ell)|}{(1 + |f'(\ell)|)} < \frac{\varepsilon}{2}. \tag{8}$$

And the definition of $\lim_{w \to 0} r_2\,(w)\,/w = 0$ implies there exists $\delta_2 > 0$ such that if $0 < |w| < \delta_2$ then

$$\left|\frac{r_2(w)}{w}\right| < \frac{\varepsilon}{2\,(1 + |g'(k)|)}, \quad \text{i.e.} \quad |r_2(w)| < \frac{\varepsilon}{2\,(1 + |g'(k)|)}\,|w|\,. \tag{9}$$

Why this most complicated factor of $(1 + |g'\,(k)|)$? It is because (9) now gives

$$\left|\frac{r_2\big(r_1(w) + wg'(k)\big)}{w}\right| < \frac{\varepsilon}{2\,(1 + |g'(k)|)} \frac{|r_1(w) + wg'(k)|}{w}.$$

We could use (8) to bound the r_1 factor in the right hand side, but instead we go back to the definition, choosing $\varepsilon = 1$ to find $\delta_3 > 0$ such that if $0 < |w| < \delta_3$ then

$$\left|\frac{r_1(w)}{w}\right| < 1, \quad \text{i.e.} \quad |r_1(w)| < |w|\,.$$

Then for $0 < |w| < \min\,(\delta_2, \delta_3)$ we have

$$\left|\frac{r_2\big(r_1(w) + wg'(k)\big)}{w}\right| < \frac{\varepsilon}{2\,(1 + |g'(k)|)} \frac{|w| + |wg'(k)|}{w} = \frac{\varepsilon}{2}, \tag{10}$$

the complicated factor has vanished!

Let $\delta = \min(\delta_1, \delta_2, \delta_3) > 0$ and assume $0 < |w| < \delta$. Then (8) and (10) combine in

$$\left|\frac{R(w)}{w}\right| \leq \left|\frac{r_2\big(r_1(w) + wg'(k)\big)}{w}\right| + \left|\frac{r_1(w)}{w} f'(\ell)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In this way we have verified the definition of $\lim_{w\to 0} R(w)/w = 0$.

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8. **Example 3.1.17** *of* ***Inverse Rule****. For $q \in \mathbb{N}$ prove that*

$$\frac{d}{dy} y^{\frac{1}{q}} = \frac{1}{q} y^{\frac{1}{q}-1},$$

for all $y > 0$.

Solution Here $g(y) = y^{1/q}$, which is the inverse function of $f(x) = x^q$ on $(0, \infty)$ (which we know has an inverse since f is strictly increasing and continuous on $(0, \infty)$). We know that $df(x)/dx = qx^{q-1}$, so

$$\frac{d}{dy} y^{\frac{1}{q}} = \frac{dg(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=g(y)}} = \frac{1}{qx^{q-1}|_{x=y^{1/q}}} = \frac{1}{q} y^{\frac{1}{q}-1}.$$

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